VI.

PROBABILITIES AT THE THREE-BALL GAME OF BILLIARDS.

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In the three-ball game of billiards, the person who makes a successful shot adds one to his counts. In case of a discount, the person who gives the discount loses, moreover, one from his count, when his opponent makès a successful shot. In the case when he gives a double discount, he loses two for each successful shot; and, in the same way, for a treble, quadruple, &c., discount, he loses three, four, &c., points from his count. In the grand discount, he loses all which he may have made, whenever his opponent succeeds in his shot. Whenever a player fails in his shot, the other player takes the cue.

Let the two players be A and B, and let h be the whole number of points of the game. Let a be the probability that A will make his shot, and b the probability that B will make his shot. No allowance is made for the increase of probability of a successful shot after the first shot, although this is a very important consideration with good players. It may justly be thought that the failure to recognize this change of probability reduces the practical value of the investigation. But imperfection is inevitable in the earlier stages of any research.

Let, then, $$A = \frac{a}{a+b-ab}, \quad B = \frac{b}{a+b-ab},$$

so that $$Ab = Ba = A + B - 1,$$

$$(1-a)\,(1-b) = 1 - \frac{a}{A} = 1 - \frac{b}{B}.$$

Let A give n discounts to B. When A needs i more points to make the game, and B needs j more points, let

$F(i,j)$ be A's probability of winning when he has the play, and

$f(i,j)$ be A's probability of winning when B has the play.

The fundamental equations are obviously

$$F(i,j) = aF(i-1,j) + (1-a)\,f(i,j),$$
$$f(i,j) = bf(i+n,j-1) + (1-b)\;F(i,j),$$

in which $i+n$ must be reduced to h whenever it exceeds h. Substitution, transposition, and division give at once

$$\begin{aligned} F(i,j) &= AF(i-1,j) + (1-A)\,f(i+n,j-1) \\ &= AF(i-1,j) + BF(i+n,j-1) - aBF(i+n-1,j-1). \end{aligned}$$

When no discount is given, this equation becomes

$$F(i,j) = AF(i-1,j) + BF(i,j-1) - aBF(i-1,j-1),$$

which is an especial case of an equation solved by Laplace in his Calculus of Probabilities.

When it is a grand discount, or whenever

$$i > h - n,$$

the equation becomes

$$\begin{aligned} F(i,j) &= AF(i-1,j) + (1-A)\,f(h,j-1) \\ &= AF(i-1,j) + BF(h,j-1) - aBF(h-1,j-1). \end{aligned}$$

These are special cases, whatever may be the discount: —

$$\begin{aligned} F(i,\ 0) &= f(i,0) = 0, \\ F(0,\ j) &= f(0,j) = 1, \\ F(1,1) &= A, \\ F(i,1) &= AF(i-1,1) = A^i. \end{aligned}$$

In the case of the grand discount,

$$\begin{aligned} F(i,j) - f(h,j-1) &= A\,[F(i-1,j) - f(h,j-1)], \\ &= A^i[1 - f(h,j-1)], \\ F(i,j) &= A^i + (1-A^i)\,f(h,j-1), \\ F(i-1,j) &= A^{i-1} + (1-A^{i-1})\,f(h,j-1). \end{aligned}$$

These equations, substituted in the first of the fundamental equations, give

$$f(i, j) = A^i(1 - b)\,[1 - f(h, j, -1)] + f(h, j - 1).$$

Let

$$C = A^h(1 - b),$$

and this equation gives

$$1 - f(h, j) = (1 - C)\,[1 - f(h, j - 1)] = (1 - C)^j$$

$$F(i, j) = A^i + (1 - A^i)\,[1 - (1 - C)^{j-1}] = 1 - (1 - A^i)\,(1 - C)^{j-1}$$

$$F(h, h) = 1 - (1 - A^h)\,(1 - C)^{h-1},$$

which is the probability that A will win at the outset, if he has the cue.

PARTICULAR EXAMPLES.

1. When the player A is an unfailing shot, we have

$$a = 1 = A, \qquad B = b$$

$$A^h = 1$$

$$F(h, h) = 1.$$

2. When B is the unfailing shot, we have

$$A = a, \qquad b = 1 = B,$$

$$C = 0, \qquad F(h, h) = A^h = a^h;$$

so that if (A)'s average run is h, he can afford to give any player whatever a grand discount, if he holds the cue.

3. When A and B are equal players, we have

$$a = b, \qquad A = B = \frac{1}{2 - a}.$$

If in this case the average run of each player is a little less than 27 points, or, more accurately, if it is 26.980, either player can venture to give the other a grand discount, if he has the cue, and if the game is 50 points.

4. Other examples for the game of 50 points are contained in the following table: the first column contains the average run of the player (A), who gives the grand discount, and the second column the average run of the player (B). The chances of victory are equal,

if the player who gives the discount holds the cue at the outset. The average run is expressed to thousandths of a point: —

(*A*)'s average run.	*B*)'s average run.
42.806	149.513
37.280	72.707
32.963	47.048
29.504	34.074
26.980	26.980
26.975	26.966
26.922	26.834
26.660	26.176
18.948	13.896
13.298	5.459
4.711	1.258